Neural Coding 2021

Eigenangles: evaluating the statistical similarity of neural network simulations via eigenvector angles

27.07.2021 | **Robin Gutzen**, Sonja Grün, Michael Denker

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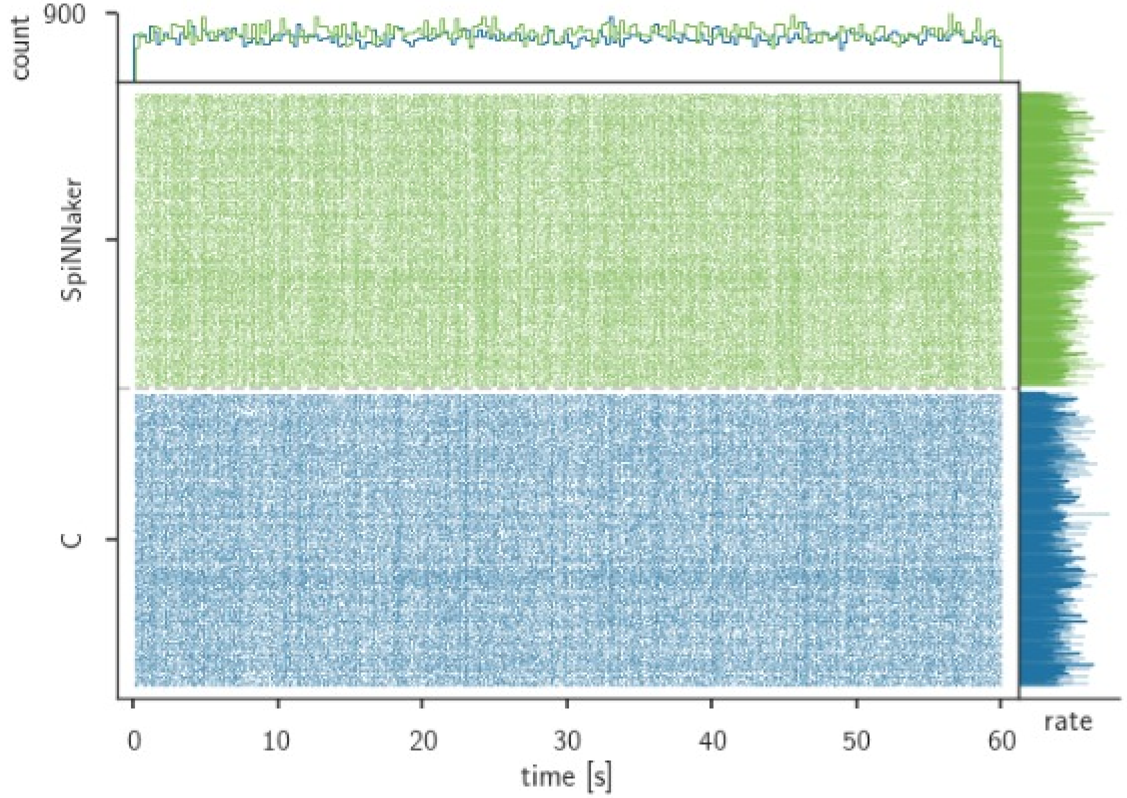
How to compare spiking network activity?



Characterization of the activity via
neuron-wise measures
(firing rate, inter-spike intervals, regularity, ...)

and statistical evaluation via
two sample tests
(Kolmogorov-Smirnov, Mann-Whitney U, effect size, ...)

form the basis for calibration and validation of models.

see *Gutzen et al. (2018)*, *Trensch et al. (2018)*, NetworkUnit

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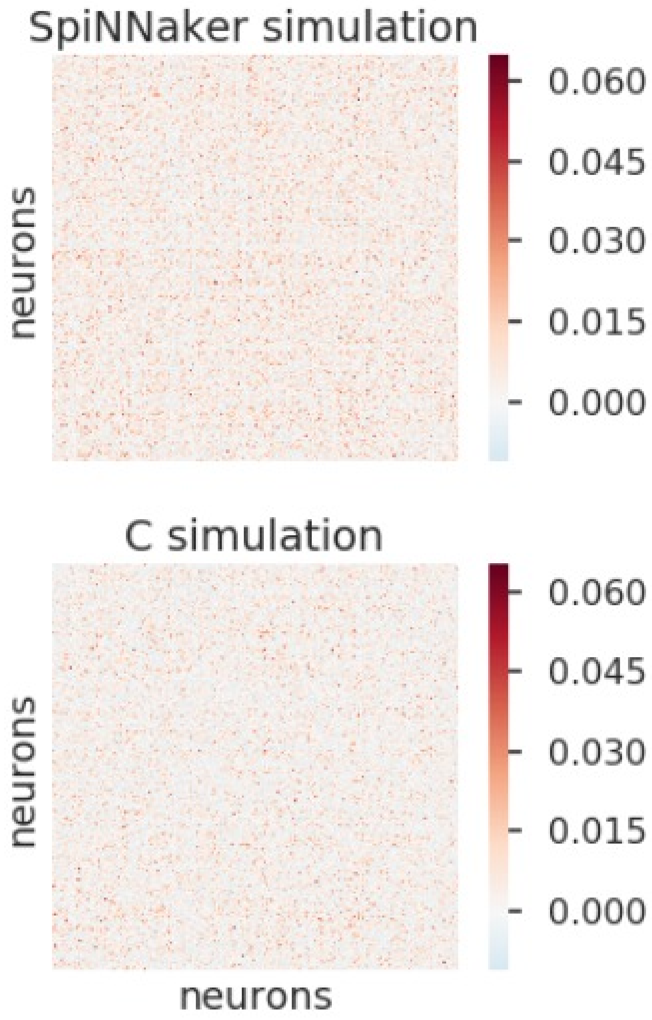
How to compare pairwise measures?



The interdependence of values in **pairwise measures** *(e.g. Pearson correlation coefficient)*

are ignored by standard two sample tests.

Instead, we compare the correlation structure via **angels between eigenvectors**

Gutzen et al. (2018), Trensch et al. (2018)

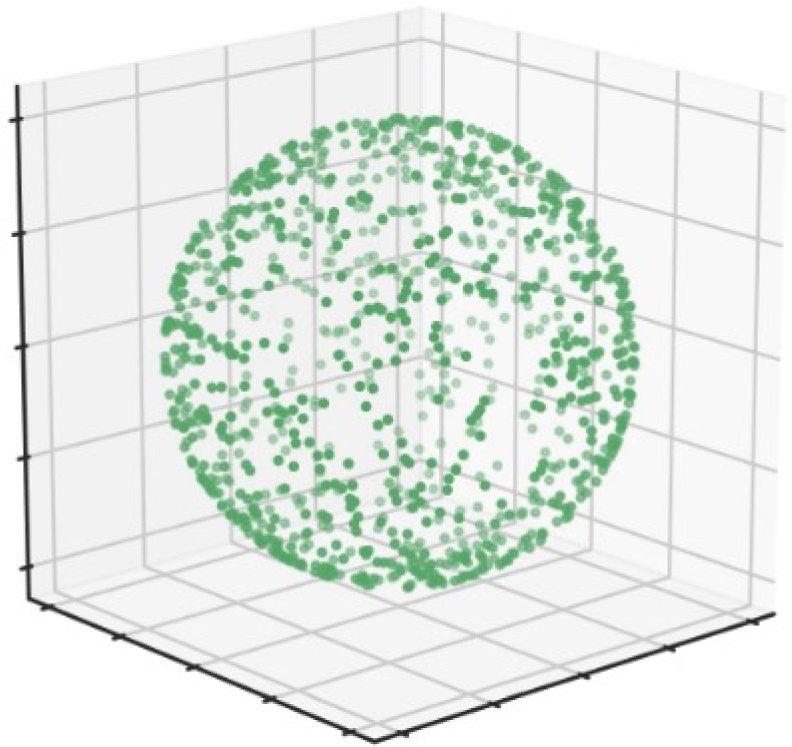
Random vectors in high dimensions

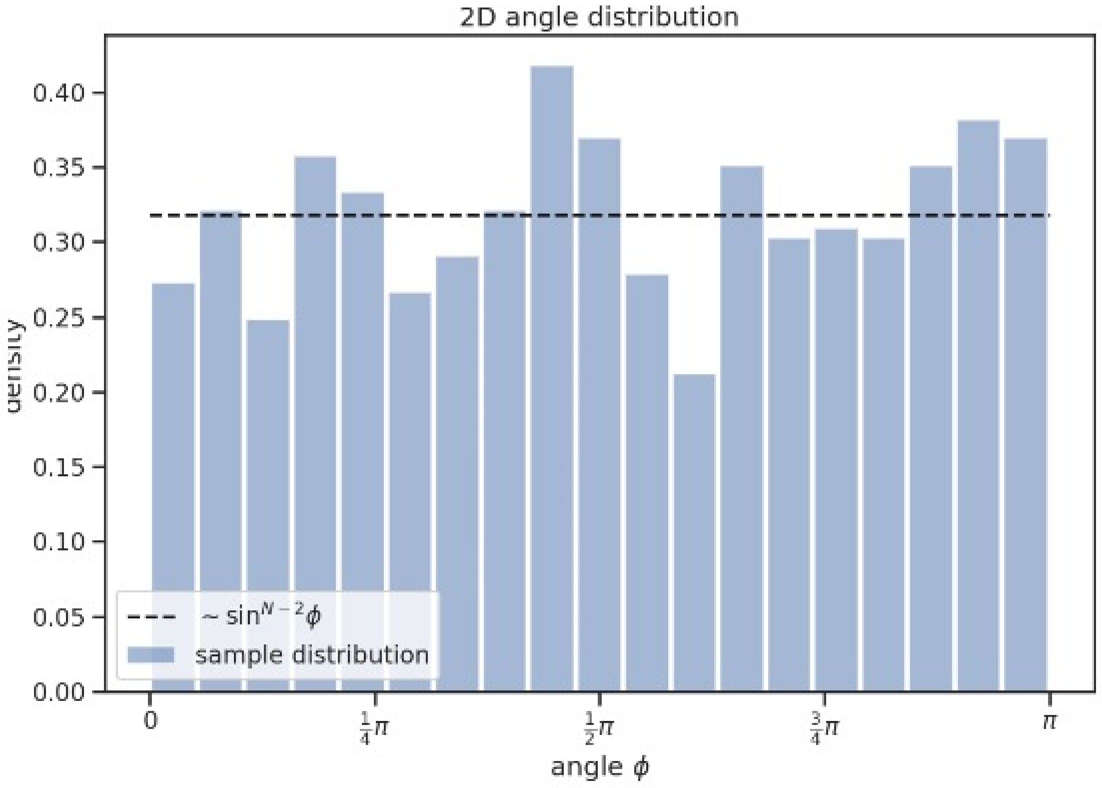
If an angle can be considered 'small' depends on the dimension

$$\phi = \arccos(\mathbf{v_1} \cdot \mathbf{v_2})$$

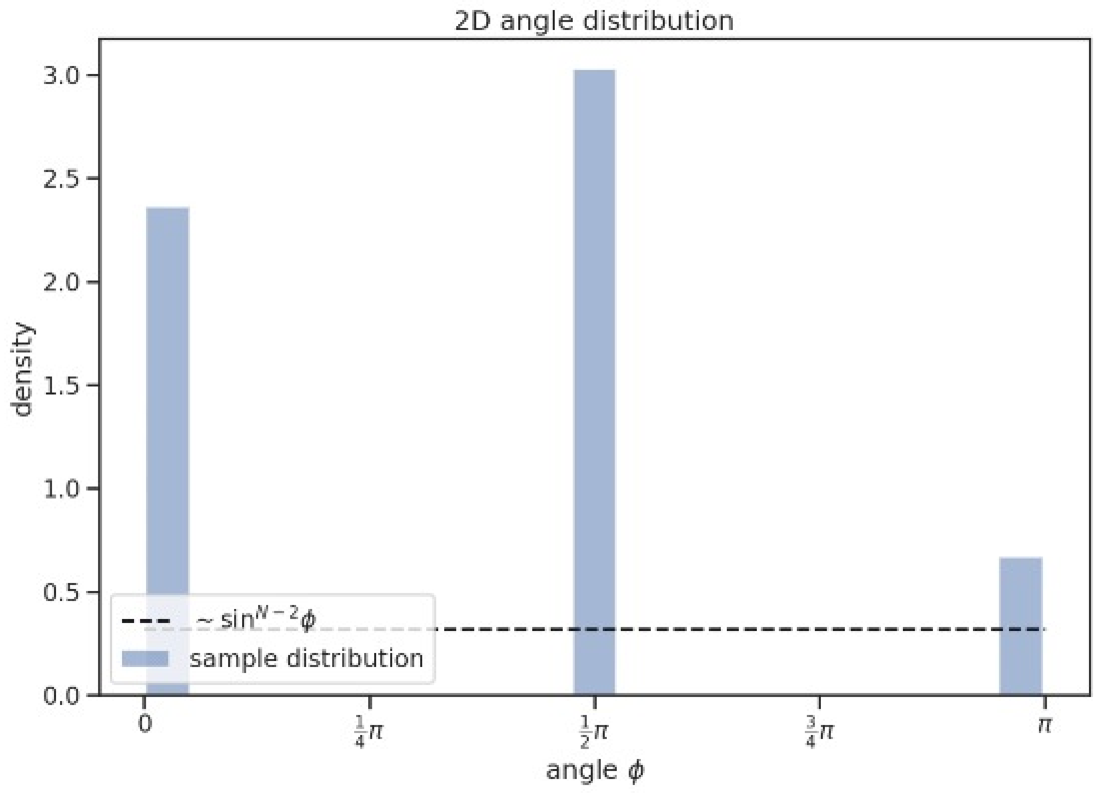
$$f(\phi) \propto \sin^{N-2}(\phi) \qquad \phi \in [0, \pi]$$

Cai et al. (2013)





Eigenvectors of random matrices behave like random vectors (dim >10)



A random correlation matrix can be created in the form

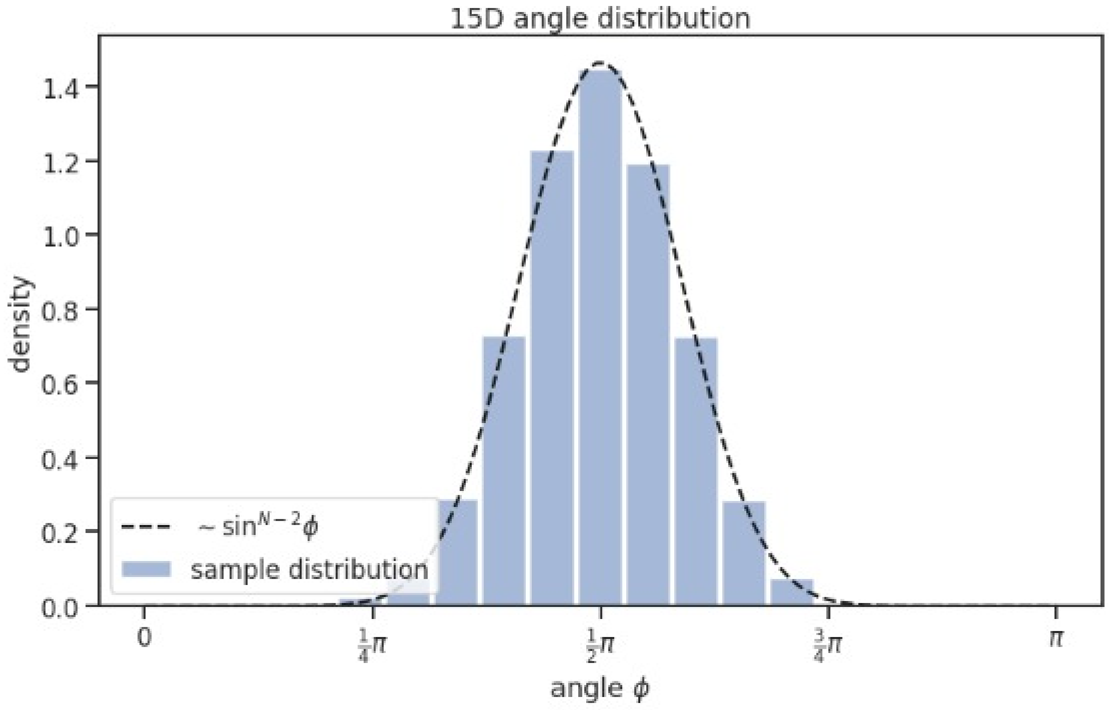
$$\mathbf{Y}_N = \mathbf{X}\mathbf{X}^T$$

where $\mathbf{X}$ is a matrix with normalized random row vectors.

Homes (1991)

Distribution of angles between random (eigen-)vectors in $\mathbb{R}^N$

$$f(\phi) \propto \sin^{N-2}(\phi) \qquad \phi \in [0, \pi]$$

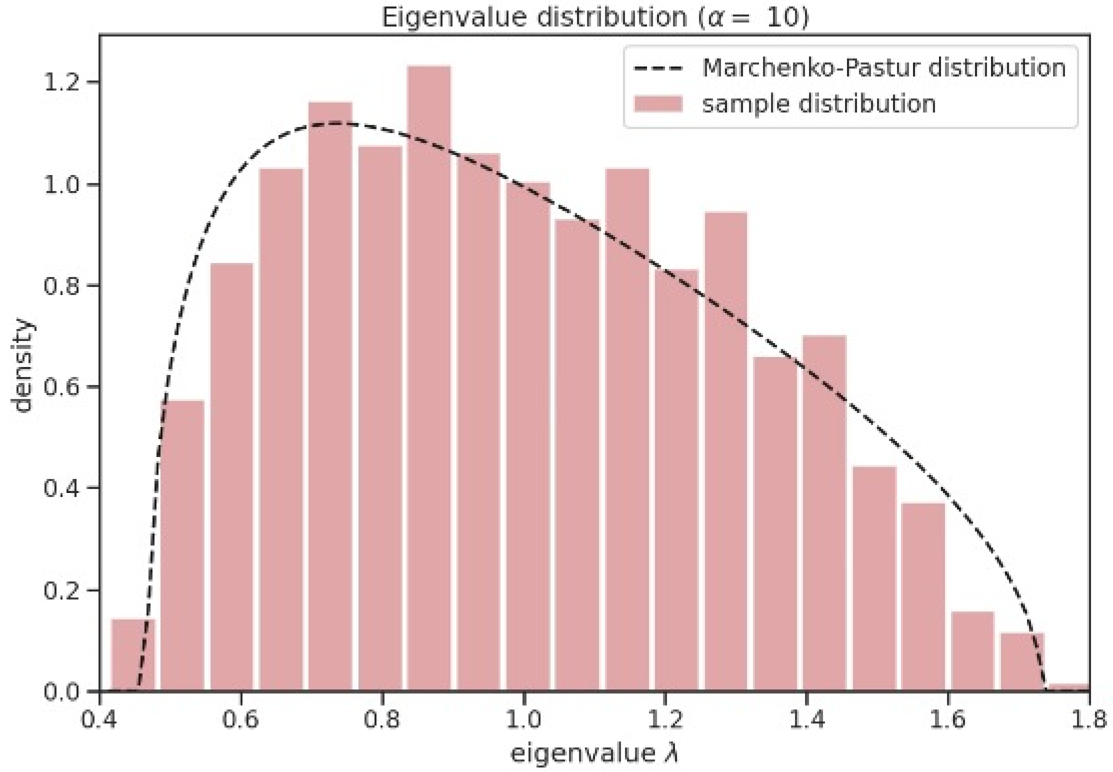


Define angle smallness as

$$\Delta = 1 - \frac{\phi}{\pi/2}$$

$$\tilde{f}(\Delta) \propto \cos^{N-2}(\Delta \cdot \pi/2) \qquad \Delta \in [-1, 1]$$

Eigenvalues indicate 'importance' of eigenvectors



$$h_\alpha(\lambda) = \frac{\alpha}{2\pi\lambda}\sqrt{(\lambda_+ - \lambda)\cdot(\lambda - \lambda_-)}$$

$$\lambda_\pm = \left(1 \pm \sqrt{\frac{1}{\alpha}}\right)^2$$

for random matrices of form

$\mathbf{Y}_N = \mathbf{X}\mathbf{X}^T$ with $\mathbf{X}$ in shape $(\alpha N) \times N$

with identically independent random entries,

with 0 mean, $\sigma^2 < \infty$ and $N \to \infty$.

Marchenko, Pastur (1967)

Random eigenvalue distribution

$$\lambda_{\pm} = \left(1 \pm \sqrt{\frac{1}{\alpha}}\right)^2$$

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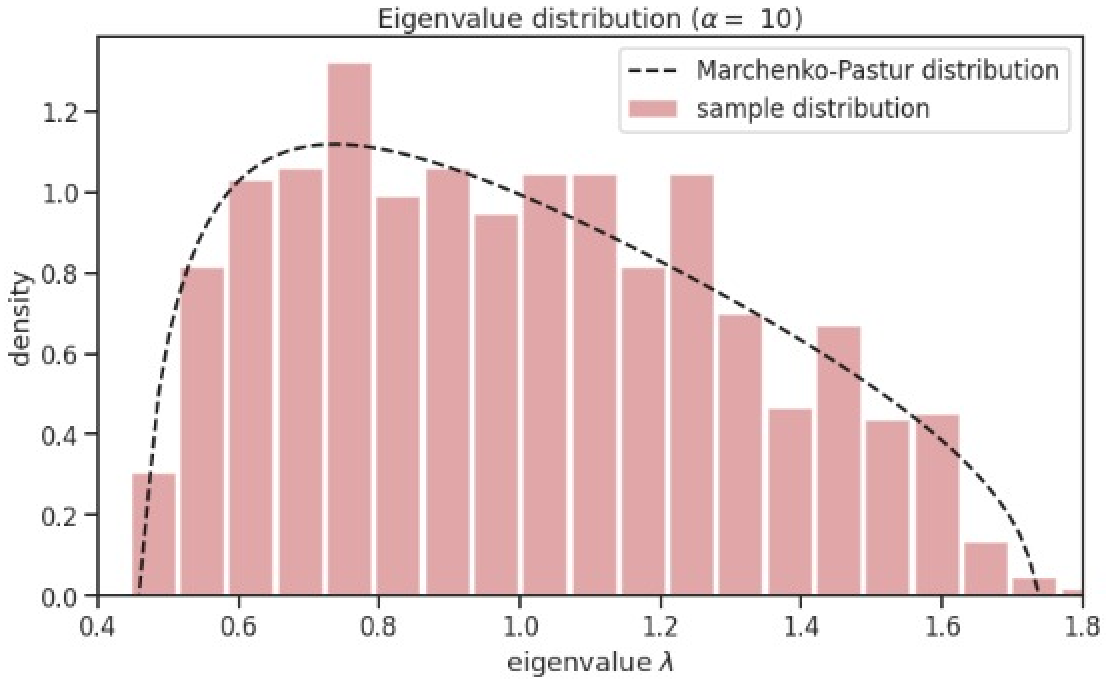

Define weights as

$$w_i \propto \sqrt{(\lambda_{A,i}^2 + \lambda_{B,i}^2)/2} \qquad \sum w_i = N$$

$$\tilde{h}_\alpha(w) \sim h_\alpha(\lambda)$$

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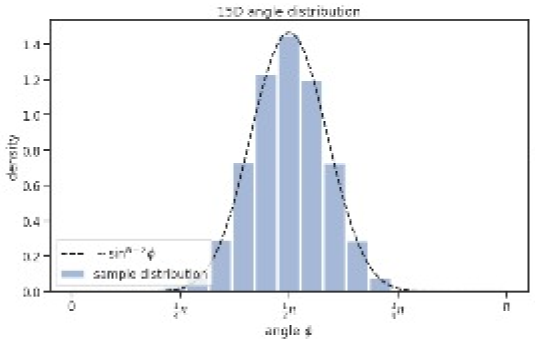

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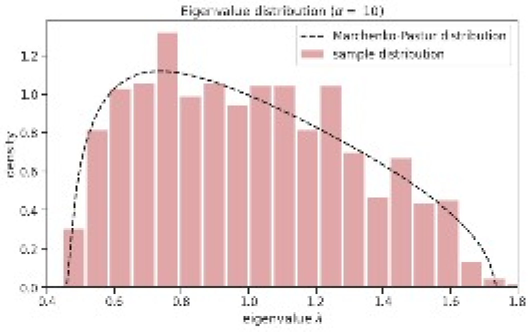

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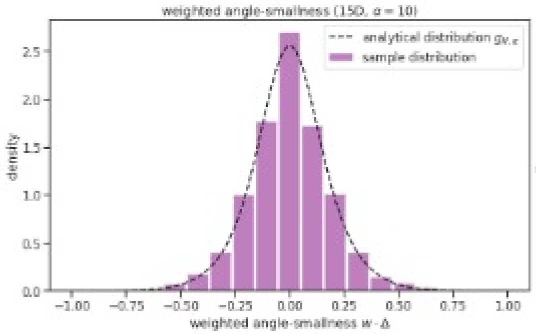
$$\tilde{h}_\alpha(w) \sim h_\alpha(\lambda)$$

Weighted angle smallness

$$g_{N,\alpha}(w\Delta) = \int_{\lambda_-}^{\lambda_+} \tilde{f}_N\left(\frac{\Delta}{\lambda}\right) \cdot h_\alpha(\lambda) \cdot \frac{d\lambda}{\lambda}$$



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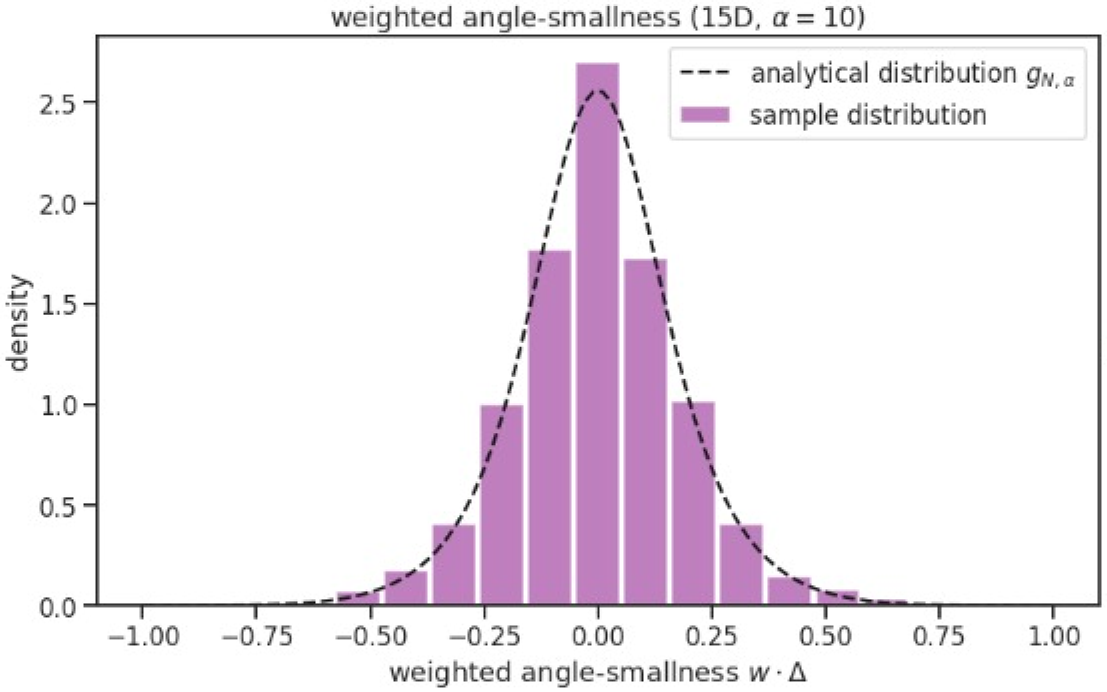

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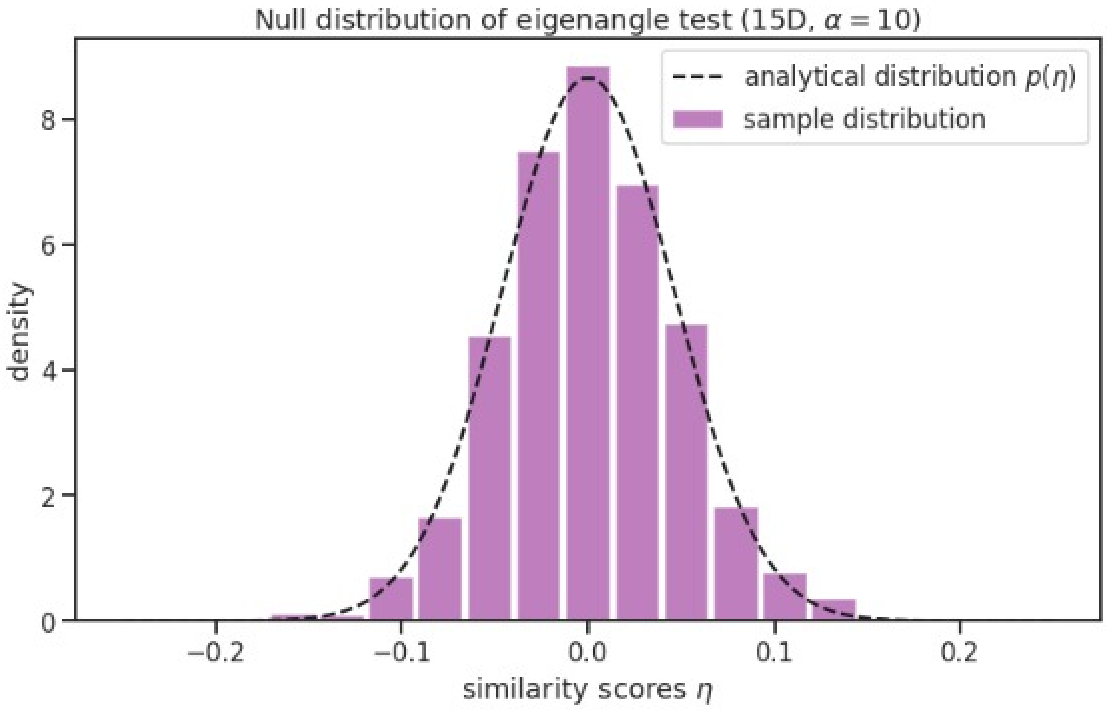
Similarity score

$$\eta = \frac{1}{N} \sum_{i}^{N} w_i \cdot \Delta_i$$

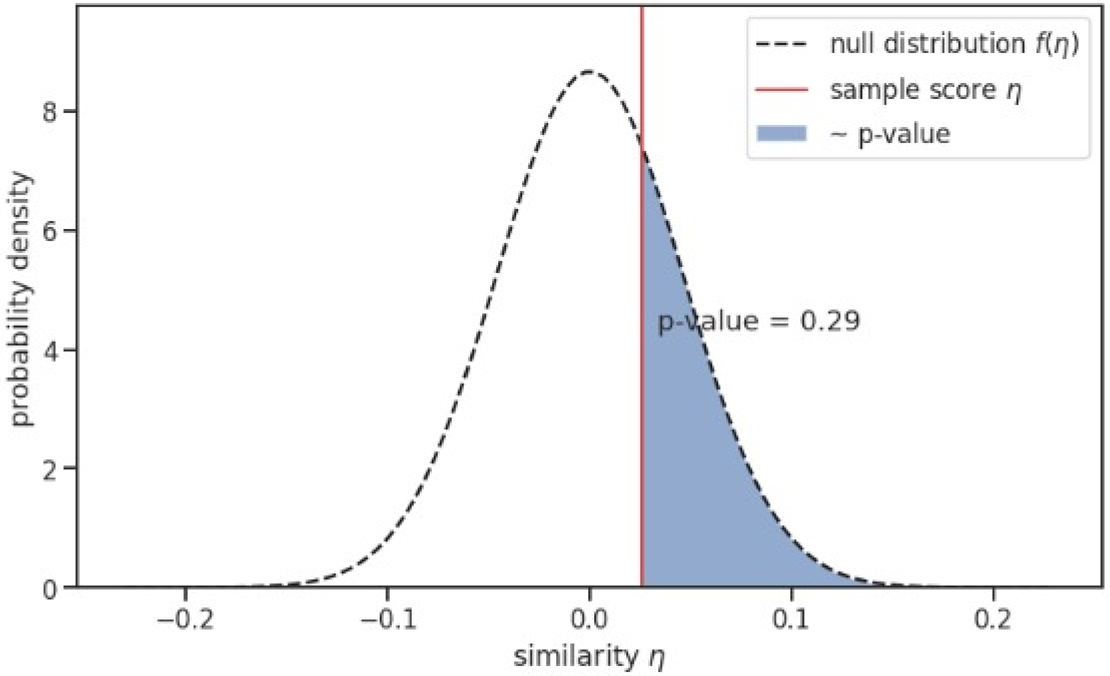


$$p(\eta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{\eta^2}{2\sigma^2})$$

$$\sigma^2 = \frac{1}{N} \int x^2 \cdot g_{N,\alpha}(x)\ dx$$

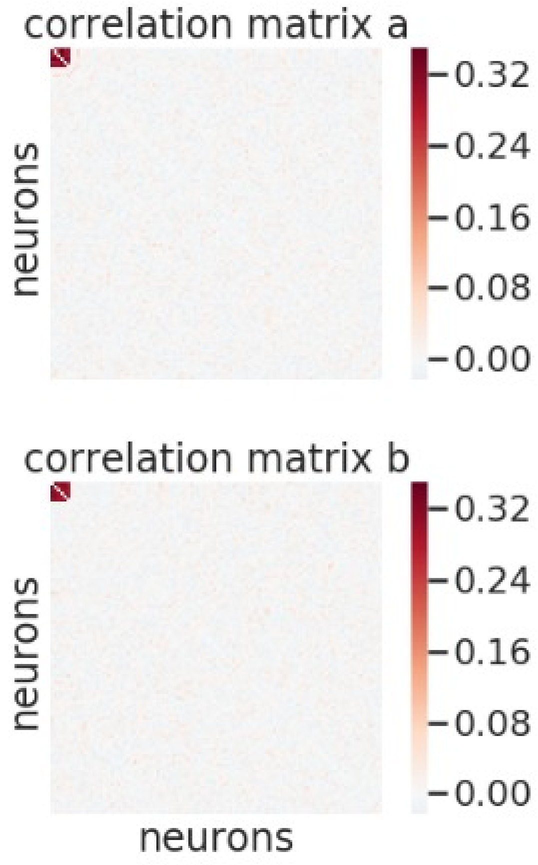
Null Hypothesis

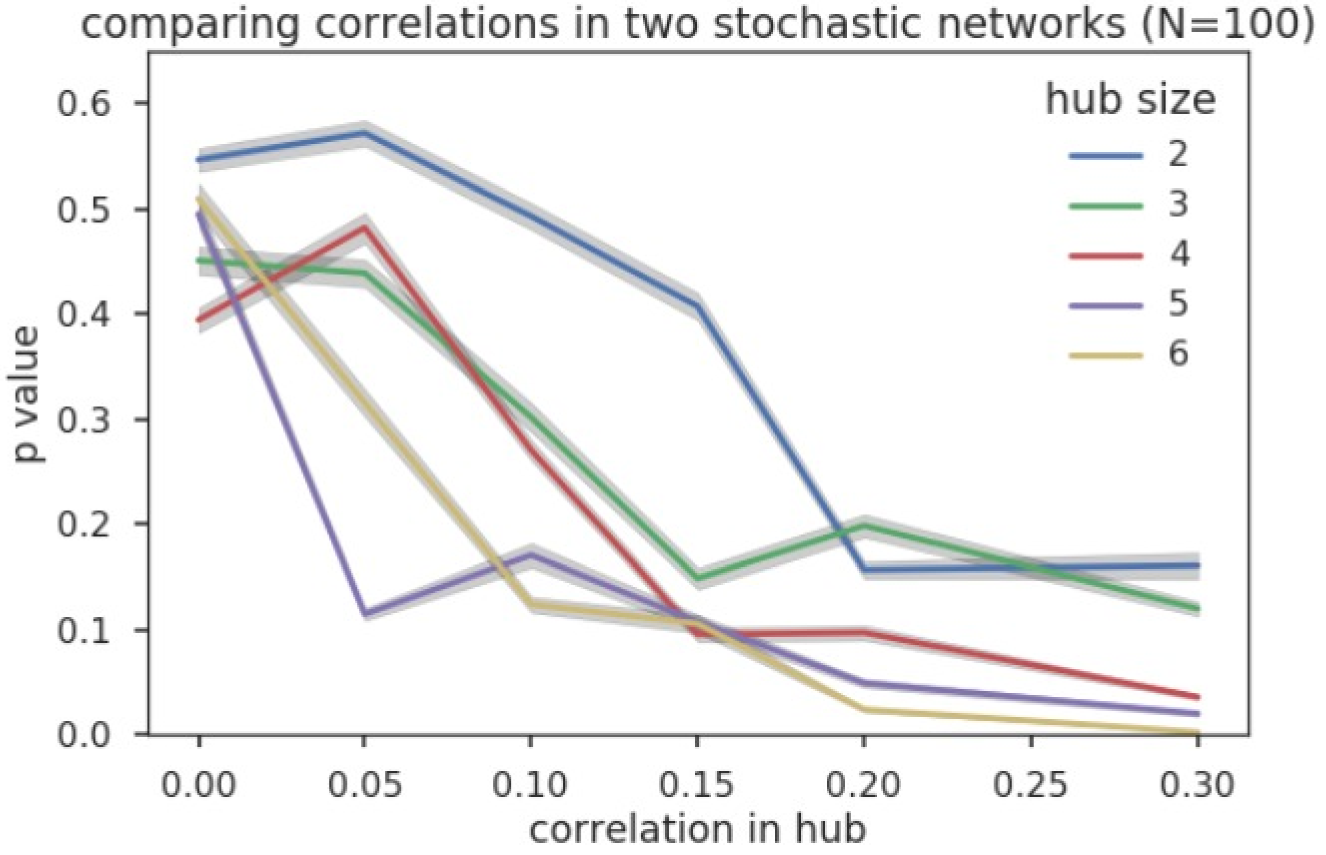
Given two independent matrices A and B of type $\mathbf{Y}_N = \mathbf{X}\mathbf{X}^T$, where $\mathbf{X}$ is a $(\alpha N) \times N$ random matrix whose entries are independent identically distributed random variables with mean 0, variance $\sigma^2 < \infty$ and $N \to \infty$.



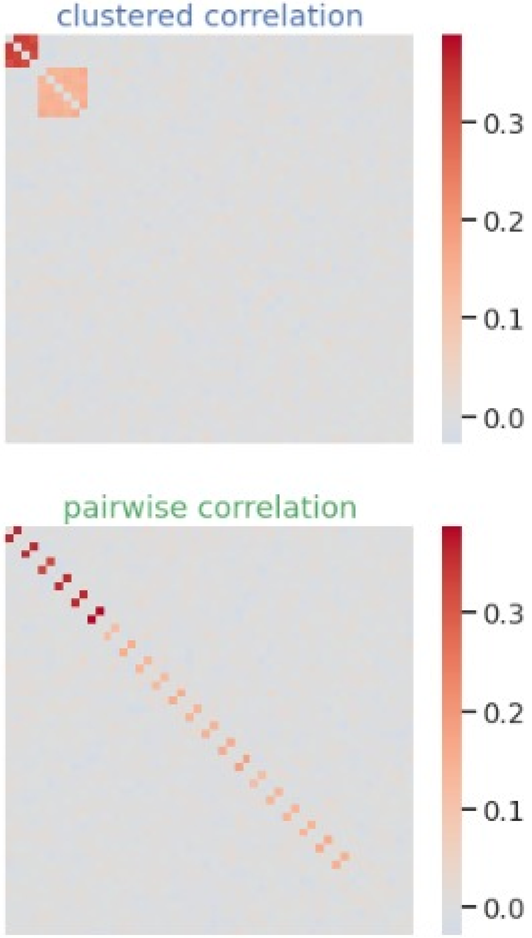
$$P = \int_{\eta}^{\infty} p(x)dx$$

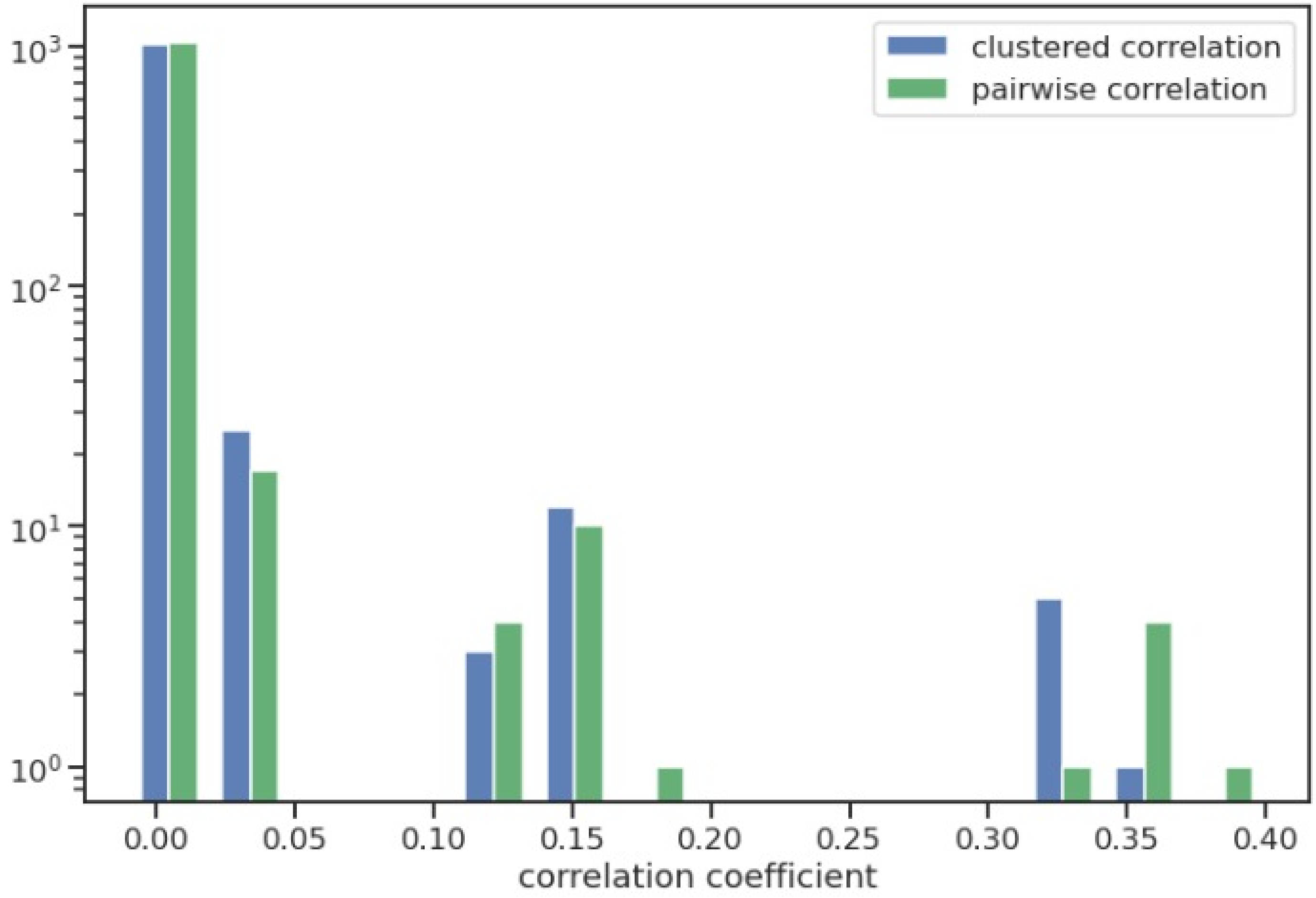
Evaluating the Eigenangle test (i)





Evaluating the Eigenangle test (ii)





Eigenangle:
p-value = 0.360 -> *dissimilar*

KS-distance:
p-value = 0.766 -> *similar*

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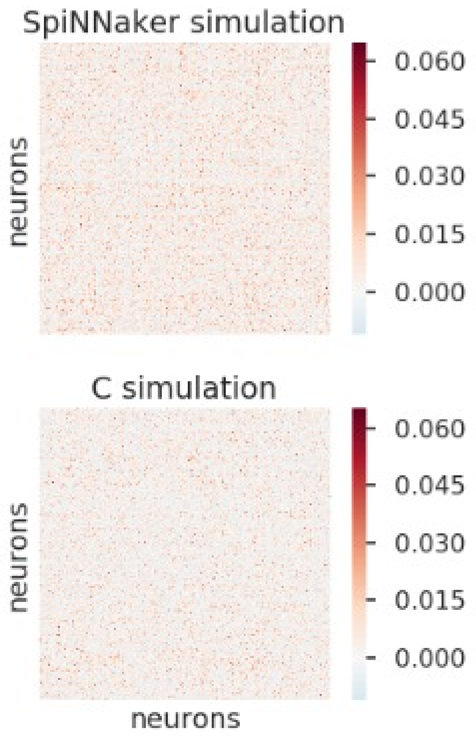
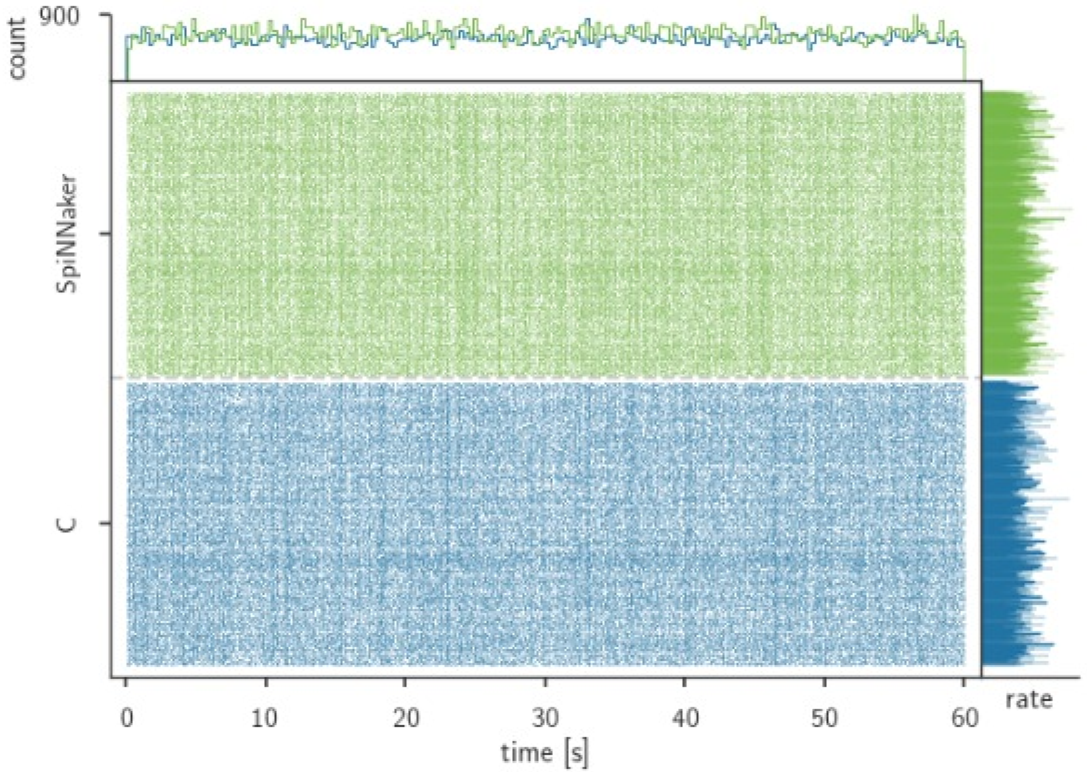
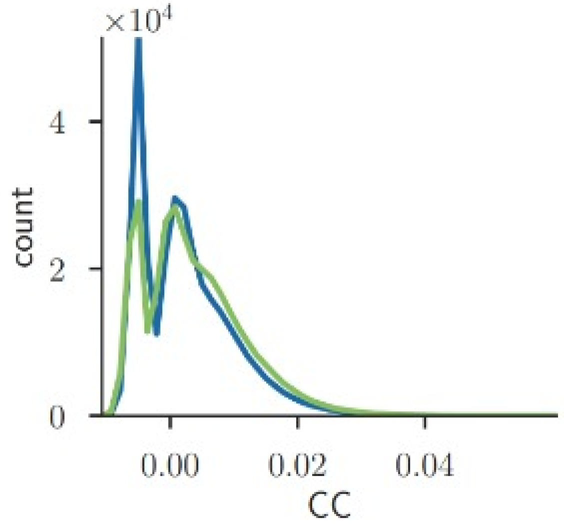
Evaluating the Eigenangle test (ii)







Eigenangle:
p-value ~ 10e-15 -> *similar*

KS-distance:
p-value ~ 0.0 -> *dissimilar*

Extension to asymmetric connectivity matrices

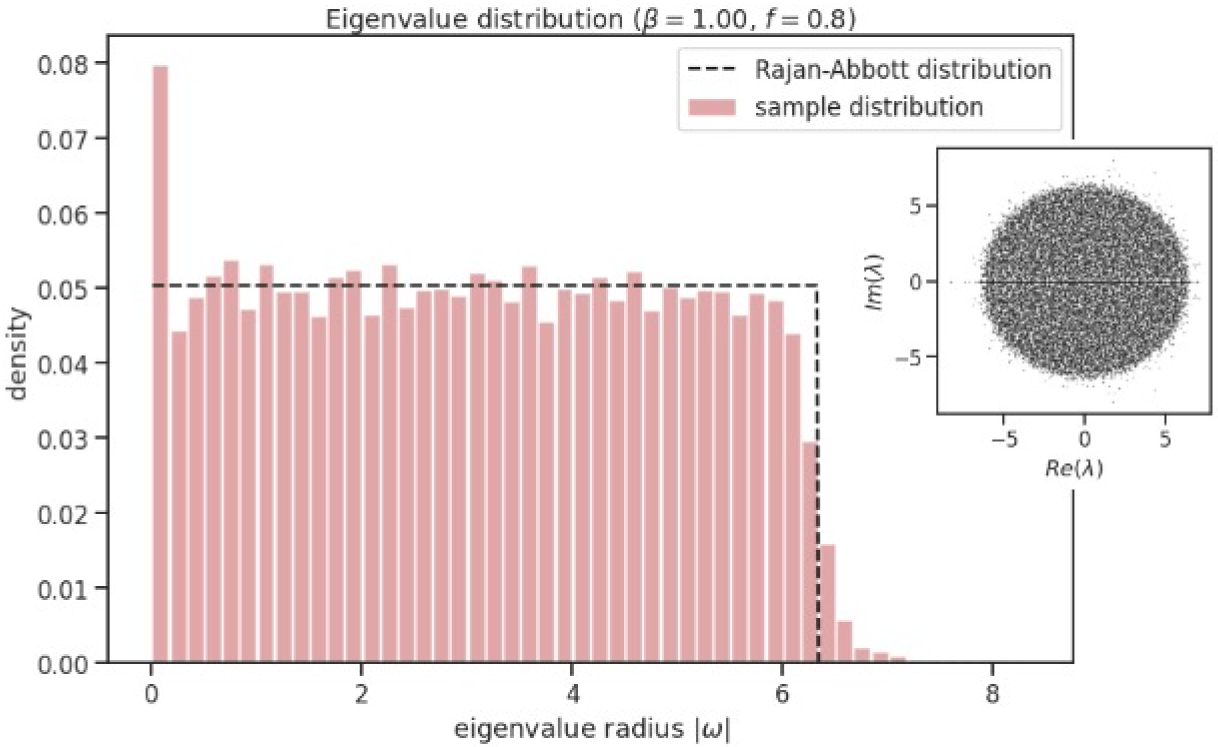
PRL **97,** 188104 (2006) PHYSICAL REVIEW LETTERS week ending 3 NOVEMBER 2006

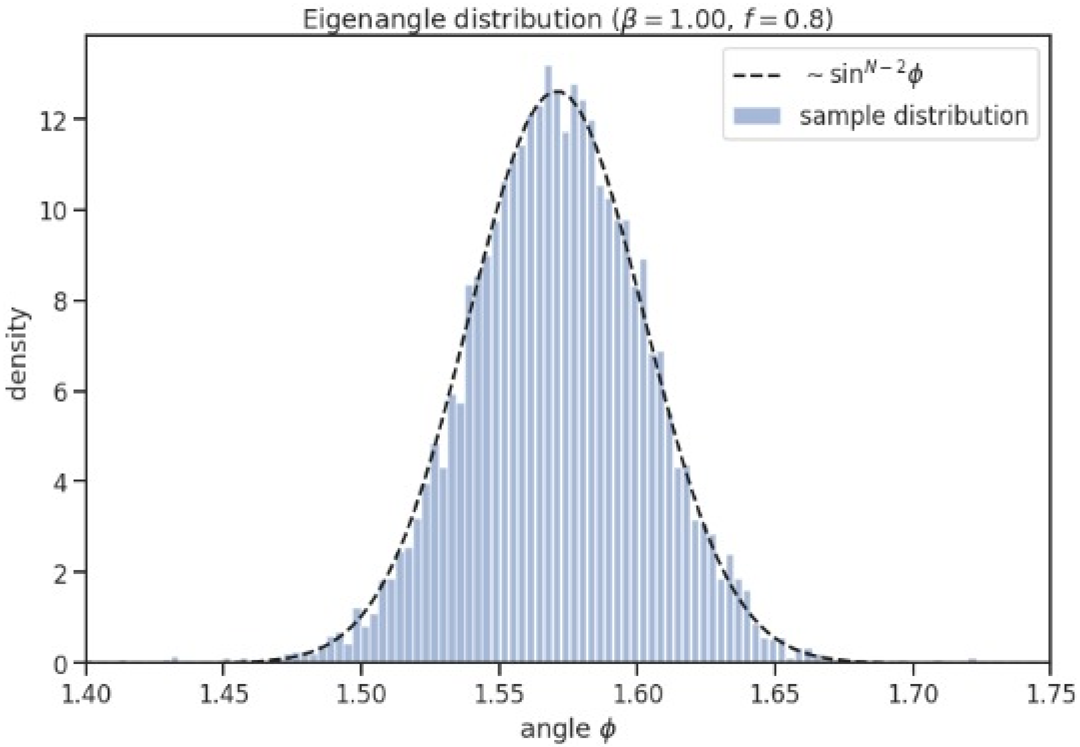
Eigenvalue Spectra of Random Matrices for Neural Networks

Kanaka Rajan and L. F. Abbott

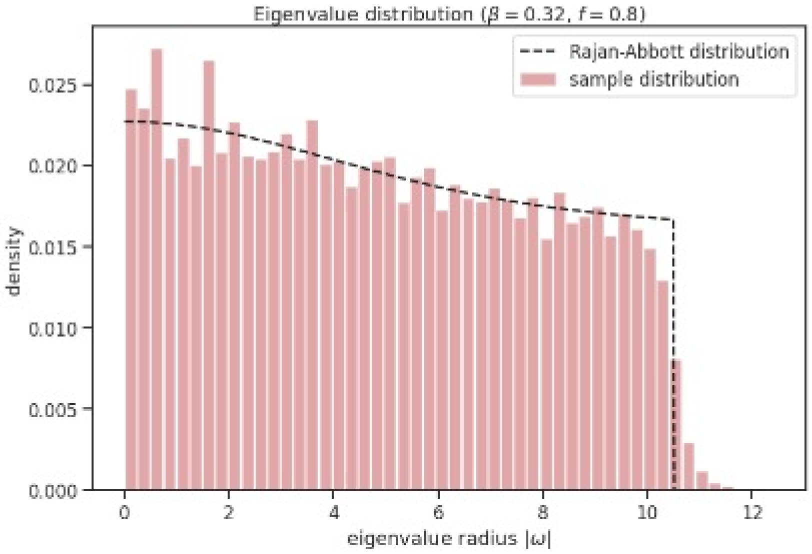
- fN exc.; $(1-f)N$ inh.
- exc. weights with μ_E; $\sigma_E^2 = \frac{1}{N}$
- inh. weights with μ_I; $\sigma_I^2 = \frac{1}{\alpha N}$
- balanced state: $f\mu_E + (1-f)\mu_I = 0$

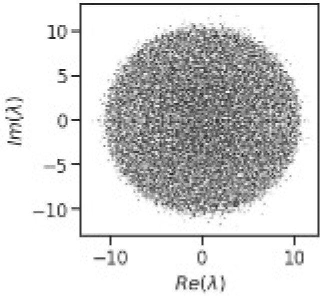
Eigenspectra of connectivity matrices

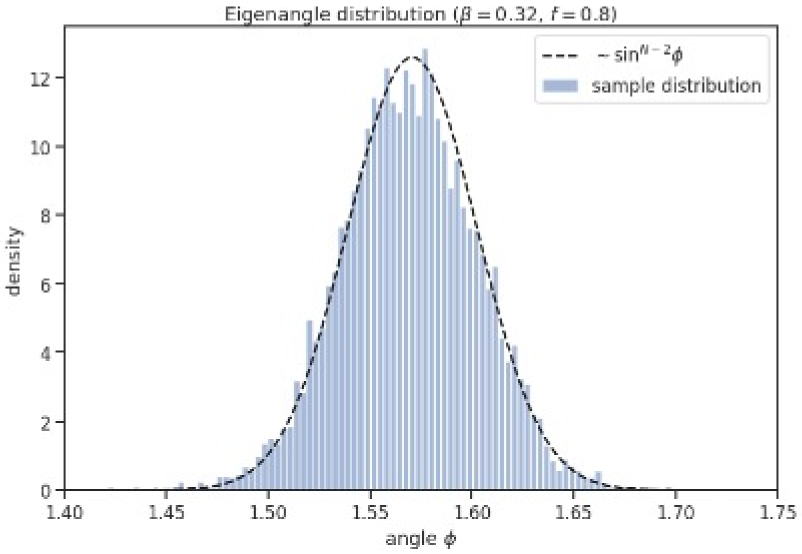


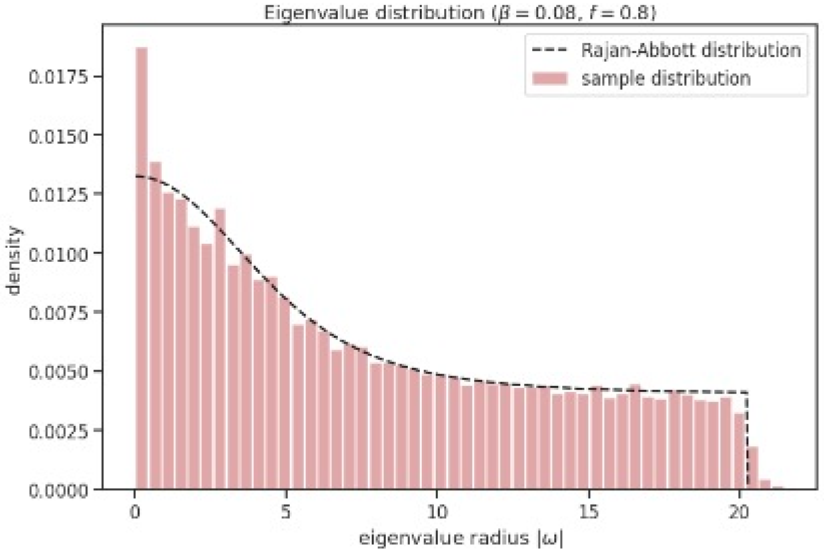


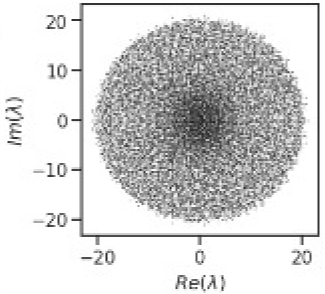
Eigenspectra of connectivity matrices

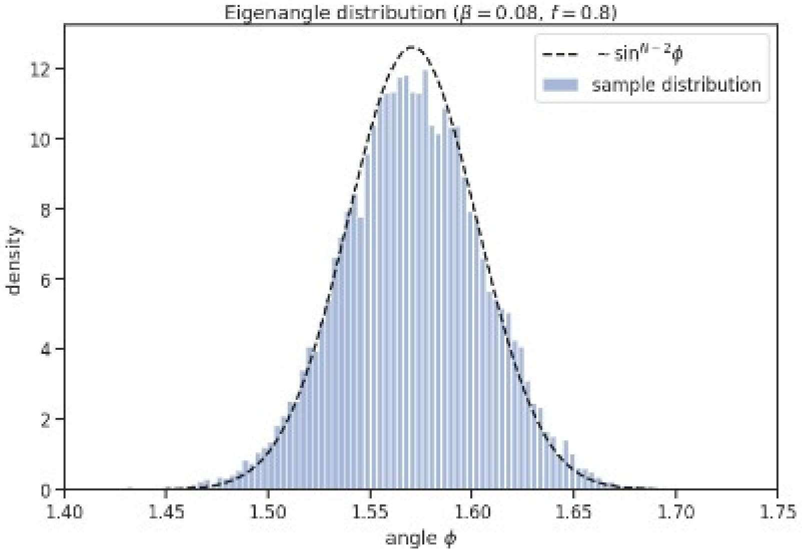












The angles between eigenvectors of matrices can detect & quantify the similarity of the correlation structures in neural network activity.

Outlook

- exploring the influence of network architectures to eigenangles
- testing to lift the limitation of neuron identities by ordering
- application to use cases (model calibration, ephys experiments)
- integration of the eigenangle test into the NetworkUnit package (v0.2)

Thank you for your interest!

@rgutzen

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