A statistical test of eigenvector angles to evaluate the similarity of neural network simulations

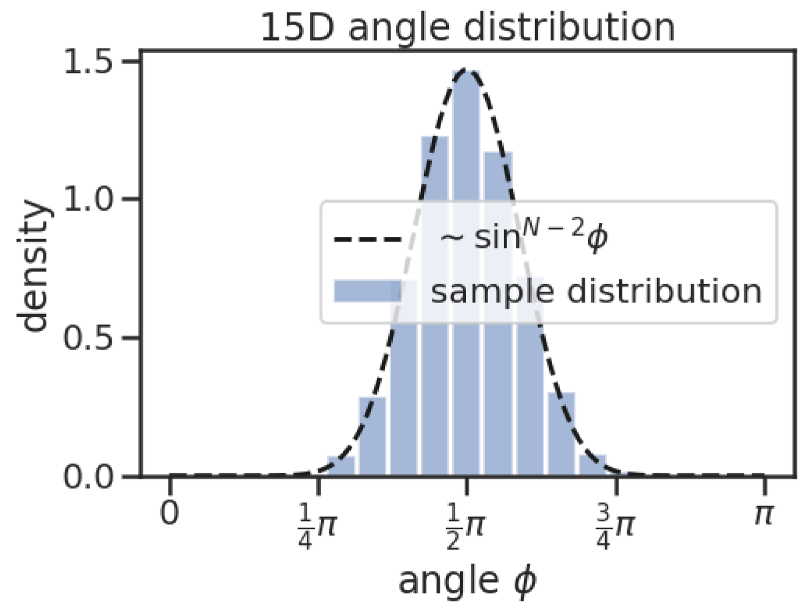
Robin Gutzen[1,2], Sonja Grün[1,2], Michael Denker[1]

[1] Institute of Neuroscience and Medicine (INM-6), Institute for Advanced Simulation (IAS-6), JARA-Intitute Brain Structure-Function Relationship (INM-10), Jülich Research Centre, Germany
[2] Theoretical Systems Neurobiology, RWTH Aachen University, Aachen, Germany

Theory

Distribution of angles between random vectors in $\mathbb{R}^N$ [1]

$$f(\phi) \propto \sin^{N-2}(\phi) \qquad \phi \in [0, \pi]$$



Null hypothesis: Given two independent matrices A and B of type $\mathbf{Y}_N = \mathbf{X}\mathbf{X}^T$, where $\mathbf{X}$ is a $(\alpha N) \times N$ random matrix whose entries are independent identically distributed random variables with mean 0, variance $\sigma^2 < \infty$ and $N \to \infty$.

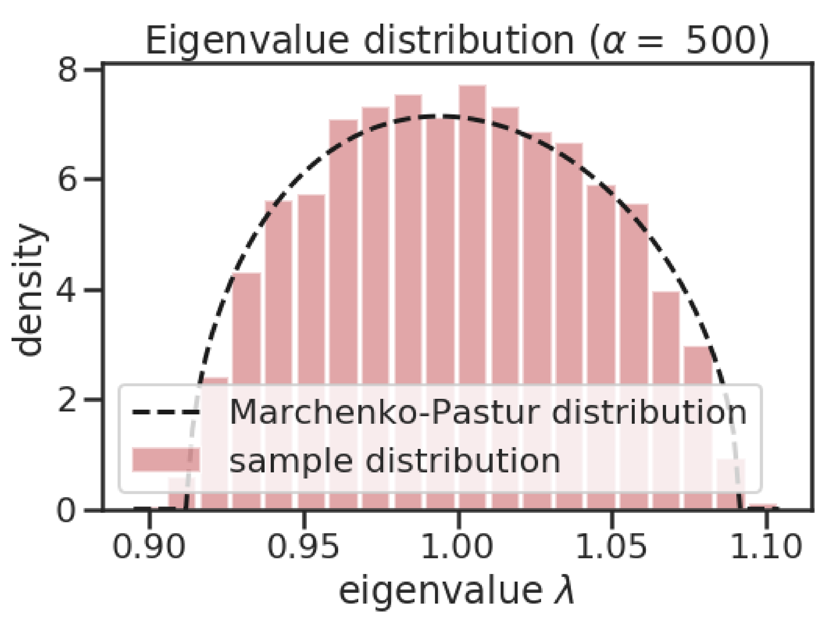
Define angle smallness as

$$\Delta = 1 - \frac{\phi}{\pi/2}$$

$$\tilde{f}(\Delta) \propto \cos^{N-2}(\Delta \cdot \pi/2) \qquad \Delta \in [-1, 1]$$

Random eigenvalue distribution [2]

$$h_\alpha(\lambda) = \frac{\alpha}{2\pi\lambda}\sqrt{(\lambda_+ - \lambda)\cdot(\lambda - \lambda_-)}$$



$$\lambda_\pm = \left(1 \pm \sqrt{\frac{1}{\alpha}}\right)^2$$

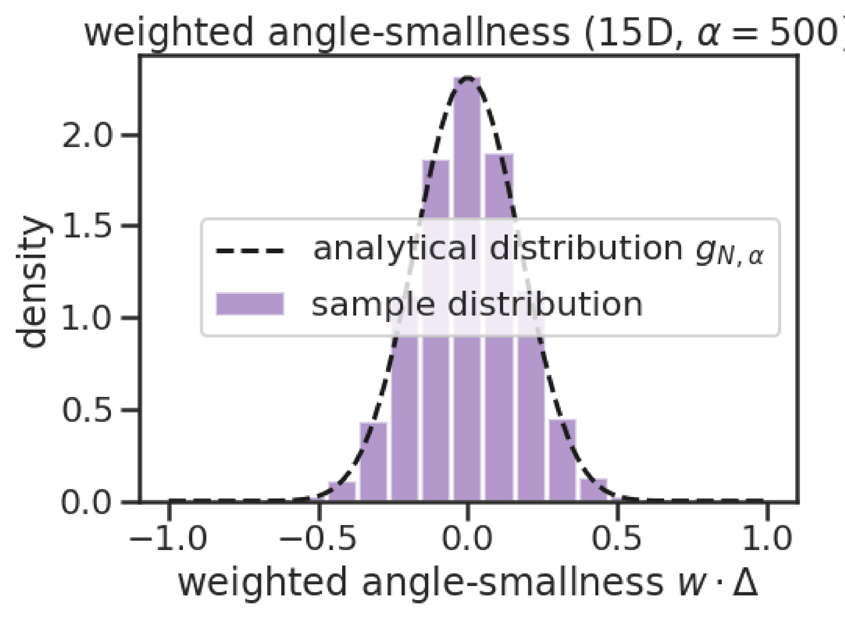
Define weights as

$$w_i \propto \sqrt{(\lambda_{A,i}^2 + \lambda_{B,i}^2)/2} \qquad \sum w_i = N$$

$$\tilde{h}_\alpha(w) \sim h_\alpha(\lambda)$$

Weighted angle smallness

$$g_{N,\alpha}(w\Delta) = \int_{\lambda_-}^{\lambda_+} \tilde{f}_N\left(\frac{\Delta}{\lambda}\right) \cdot h_\alpha(\lambda) \cdot \frac{d\lambda}{\lambda}$$



Define score as

$$\eta = \frac{1}{N}\sum_i^N w_i \cdot \Delta_i$$

Null distribution:

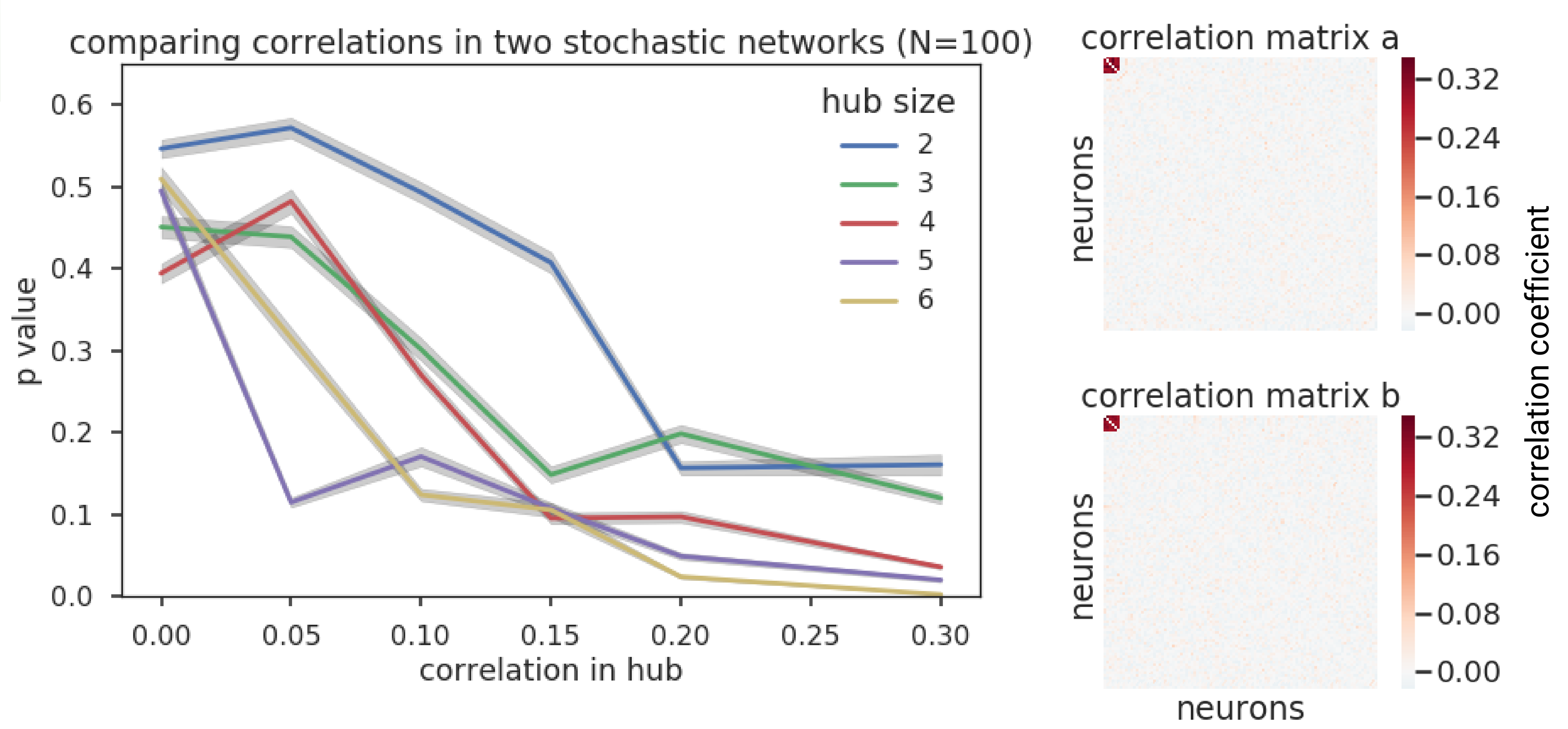
$$p(\eta) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{\eta^2}{2\sigma^2}\right) \qquad \sigma^2 = \frac{1}{N}\int x^2 \cdot g_{N,\alpha}(x)\, dx$$

Are two networks similar?

A novel statistical test based on random angles and eigenvalues can answer that.

interactive notebook → [5]

Application
for stochastic network activity

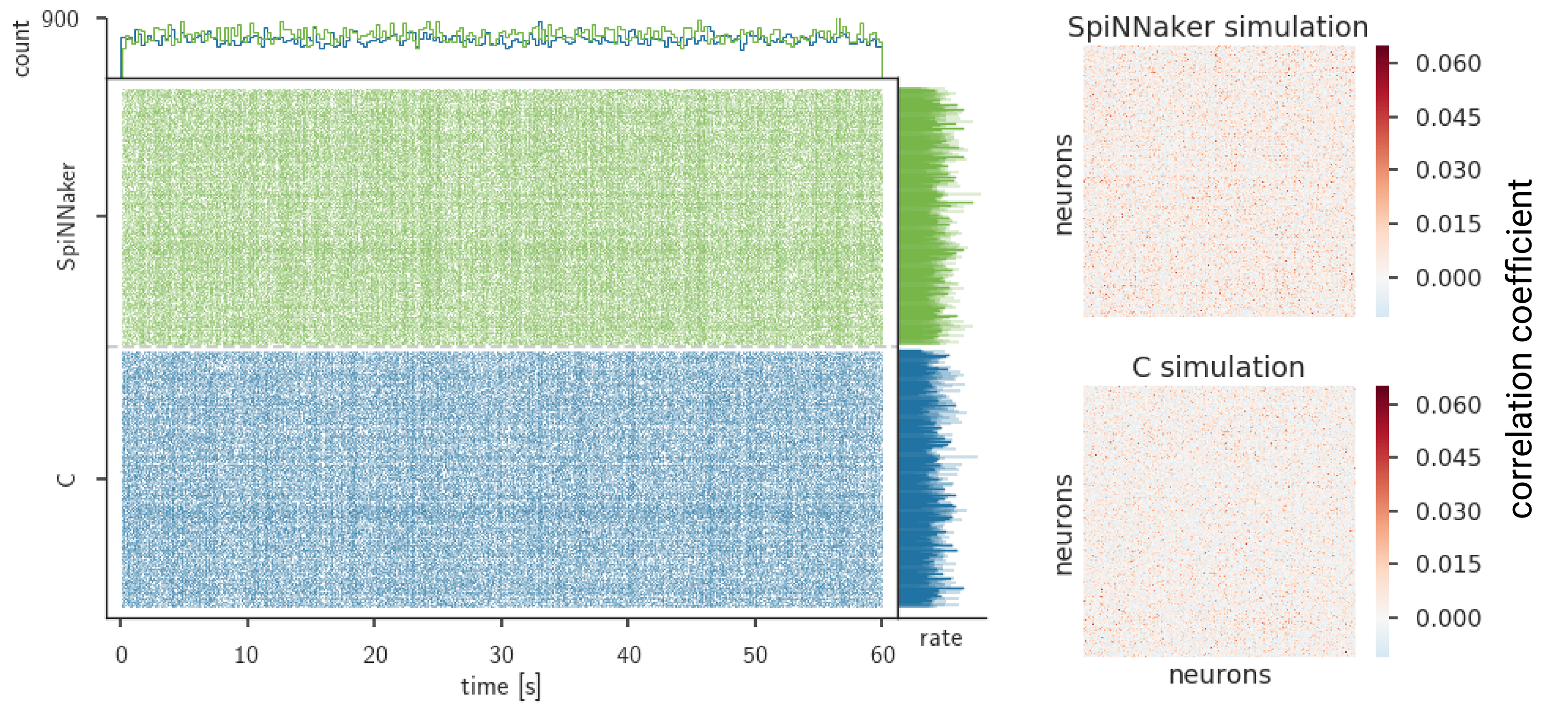


The test can detect ($p < 0.05$) isolated hubs in 100-neuron networks of size > 4 and correlation > 0.2.

Application
for simulator comparison (see [3] and [4])

Simulations of the polychonization network model (N=800)

Comparing SpiNNaker and C simulator → **p-value ~ 10e-15**



The test doesn't necessarily rely on a prevalent structure to detect similarity.

References

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[3] Trensch, et al. (2018) *Rigorous neural network simulations: a model substantiation methodology for increasing the correctness of simulation results in the absence of experimental validation data*. Frontiers in Neuroinformatics 12:81
[4] Gutzen, R., et al. (2018) *Reproducible neural network simulations: statistical methods for model validation on the level of network activity data.* Frontiers in Neuroinformatics 12:90
[5] https://mybinder.org/v2/gh/INM-6/NetworkUnit/eigenangle_demo?filepath=examples%2Feigenangle_score.ipynb

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